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Vector Equilibrium Problems. Existence Theorems and Convexity of Solution Set

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Abstract. The natural quasi-concavity of set-valued mappings in an ordered vector space is introduced. Existence theorems for vector equilibrium problems involving set-valued monotone mappings are obtained and the convexity of the solution set is shown.

Key words: Convexity, Monotonicity, Vector equilibrium problem, Set-valued mapping

1. Introduction

Let X be a real topological vector space, $D \subset X$ a nonempty subset, and let $f : D \times D \longrightarrow R$ be a given real function with $f(x, x) = 0$ for all $x \in D$. The equilibrium problem consists in finding

$$x \in D \text{ such that } f(x, y) \geqslant 0 \text{ for all } \quad y \in D.$$

This problem contains as special cases for instance, optimization problems, problems of Nash equilibria, fixed point problems, variational inequalities and complementarity problems (see [4, 7]). Recently, equilibrium problems involving vector mappings in ordered vector spaces are considered by many authors (see [1, 3, 6, 7, 10, 11], and references therein).

Let Y be a real topological vector space, $C \subset Y$ a closed, pointed and convex cone with apex at the origin and with nonempty interior, $int\, C \neq \emptyset$. Let $G, H : D \times D \longrightarrow 2^Y$ be set-valued mappings. We consider the following Vector Equilibrium Problems (in short, VEP):

(VEP 1) Find $x \in D$ such that $G(x, y) + H(x, y) \subset Y \backslash (-\mathrm{int}\, C)$, $\forall y \in D$;

(VEP 2) Find $x \in D$ such that $[G(x, y) + H(x, y)] \bigcap (Y \backslash (-\mathrm{int}\, C)) \neq \emptyset$, $\forall y \in D$;

(VEP 3) Find $x \in D$ such that $G(x, y) + H(x, y) \subset Y \backslash (-C \setminus \{0\})$, $\forall y \in D$.

Obviously, any solution of (VEP 1) is also a solution of (VEP 2). If G and H are single-valued mappings, then problems (VEP 1) and (VEP 2) both collapse to the following VEP:

(VEP 4) Find $x \in D$ such that $G(x, y) + H(x, y) \nsubseteq -int\, C$, $\forall y \in D$;
The (VEP 3) becomes (VEP 5):

(VEP 5) Find $x \in D$ such that $G(x, y) + H(x, y) \nsubseteq -C \backslash \{0\}$, $\forall y \in D$.
The (VEP 4) and (VEP 5) have been studied by Tan and Tinh [11]. Their existence results extend some main results of Blum and Oettle ([4, theorem 1 and theorem 1A]) to vector mappings. Tan and Tinh have also given some applications to efficient points, Nash equilibrium and variational inequalities.

The purpose of the paper is to discuss existence results of (VEP 1) and (VEP 3) and the convexity of the solution set. Our existence theorems extend main results of [4, 11] to vector set-valued mappings.

2. Preliminaries

Let Y be real locally convex Hausdorff space, and $C \subset Y$ a pointed, closed convex cone with apex at the origin and with int $C \neq \emptyset$. We say that the cone C satisfies the condition

(Δ) if there is a pointed, closed convex cone $\widetilde{C}$ such that $C \setminus \{0\} \subset \text{int } \widetilde{C}$.
It is well-known that if C has a base , then C satisfies the condition (Δ) (see [7], p. 234).The cone

$$C^* = \{f \in Y^* : f(x) \geqslant 0, \forall x \in C\}$$

is called the dual cone of C, where Y^* is the topological dual of Y. C^* is a convex cone. The set

$$C^\sharp = \{f \in Y^* : f(x) > 0, \forall x \in C \backslash \{0\}\}$$

is called the quasi-interior of C^*. If $C^\sharp \neq \emptyset$, then $C^\sharp \cup \{0\}$ is a nontrivial convex cone. It is well-known that $C^\sharp \neq \emptyset$ if and only if C has a base (see [8]).

DEFINITION 1 ([5]). Let X and Y be topological spaces, $T: X \longrightarrow 2^Y$ a set-valued mapping.

(i) T is said to be upper semi-continuous (in short, u.s.c.) at $x \in X$ if, for each open set $V \subset T(x)$, there is a neighbourhood U of x such that for each $z \in U$, $T(z) \subset V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) T is said to be lower semi-continuous (in short, l.s.c.) at $x \in X$ if, for each open set V with $T(x) \cap V \neq \emptyset$, there is a neighbourhood U of x such that for each $z \in U$, $T(z) \cap V \neq \emptyset$. T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) T is said to be closed if the graph $G_r(T)$ of T, $G_r(T) = \{(x, y) : x \in X,\ y \in T(x)\}$, is closed in $X \times Y$.

By the definition, it is easy to show the following lemma.

LEMMA 1([6]). *Let X, Y and T be as in Definition 1. Then*

(i) *T is closed if and only if for any net* $\{x_\alpha\}$ *in* X, $x_\alpha \to x$ *and any net* $\{y_\alpha\}$, $y_\alpha \in T(x_\alpha)$, $y_\alpha \to y$, *one has* $y \in T(x)$.

(ii) *T is l.s.c. at* $x \in X$ *if and only if for any* $y \in T(x)$ *and any net* $\{x_\alpha\}, x_\alpha \to x$, *there is a net* $\{y_\alpha\}$ *such that* $y_\alpha \in T(x_\alpha), y_\alpha \to y$.

(iii) *If Y is compact, then T is u.s.c. if and only if T is closed.*

DEFINITION 2([9]). Let X and Y be real locally convex spaces, $C \subset Y$ a closed convex cone with apex at the origin, and $D \subset X$ a nonempty subset. Let $T : D \longrightarrow 2^Y$ be a set-valued mapping. T is said to be lower semi-continuous with respect to C at $x \in D$ (in short, C-l.s.c.) if for any $y \in T(x)$ and each neighbourhood V of y, there is a neighbourhood U of x such that for each $z \in U \cap D$, $T(z) \cap (V + C) \neq \emptyset$. T is said to be C-l.s.c. on D if it is C-l.s.c. at all $x \in D$.

REMARK 1. (i) If T is l.s.c. at $x \in D$, then it is C-l.s.c. at $x \in D$; (ii) If T is single-valued and (-C)-l.s.c. at $x \in D$, then it is C-u.s.c. at $x \in D$ in the sense of Tan and Tinh ([11, Definition 2.6]).

LEMMA 2. *Let D, Y and C be as in Definition 2, and* $T : D \longrightarrow 2^Y$ *be C-l.s.c. on D. Then the set* $A = \{x \in D : T(x) \subset Y \backslash \text{int } C\}$ *is closed in D.*

Proof. We can suppose that int $C \neq \emptyset$. Let $x \in D$ and a net $\{x_\alpha\}$ in A such that $x_\alpha \to x$. We need to show $x \in A$. If $x \notin A$, then $T(x) \not\subset Y \backslash \text{int } C$. Hence, there is a point $y \in T(x)$ such that $y \in \text{int } C$. Since int C is a neighbourhood of y and T is C-l.s.c., there is a neighbourhood U of x such that $\forall z \in U \cap D$,

$$T(z) \cap (\text{int } C + C) = T(z) \cap \text{int } C \neq \emptyset \tag{1}$$

Since $x_\alpha \to x$, there is a β such that $\forall \alpha \geqslant \beta$, $x_\alpha \in U \cap D$. Therefore, by (1), $T(x_\alpha) \cap \text{int } C \neq \emptyset$, i.e., $x_\alpha \notin A$, a contradiction. □

LEMMA 3. *Let D,Y and C be as in Lemma 2, and let* $T : D \longrightarrow 2^Y$ *be given. For any fixed* x, $y \in D$, *let* $g(t) = T(ty + (1 - t)x)$, $t \in [0, 1]$. *Assume that* $g(t)$ *is (-C)-l.s.c. at* $t = 0$, *and* $\forall t \in (0, 1], g(t) \subset Y \backslash (-\text{int } C)$. *Then* $g(0) \subset Y \backslash (-\text{int } C)$.

Proof. If the conclusion is false, then there is a point $w \in g(0)$ such that $w \in -\text{int } C$. Since $g(t)$ is (-C)-l.s.c. at $t = 0$, there is a $\delta \in (0, 1)$ such that $\forall t \in [0, \delta]$, $g(t) \cap (-\text{int } C - C) = g(t) \cap (-\text{int } C) \neq \emptyset$. This contradicts the assumption $g(t) \subset Y \backslash (-\text{int } C)$. □

DEFINITION 3([4]). Let K and D be nonempty convex subsets of a vector space X with $K \subset D$. The set

$$\mathrm{core}_D K = \{a \in K : \quad K \cap (a, y] \neq \emptyset, \quad \forall y \in D \backslash K\}$$

is called the core of K relative to D, where $(a, y] = \{x \in X : x = (1-t)a + ty, t \in (0, 1]\}$.

Motivated by Tanaka [12], we introduce the following concept of natural quasi-C-concavity for set-valued mappings.

DEFINITION 4. Let X and Y be real topological vector spaces, $D \subset X$ a convex subset, and $C \subset Y$ a convex cone. Let $F : D \longrightarrow 2^Y$ and $G : D \times D \longrightarrow 2^Y$ be given.

(i) F is said to be C-convex if for any $x, y \in D$, $t \in [0, 1]$, $tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + C$;

(ii) F is said to be C-concave if for any $x, y \in D$, $t \in [0, 1]$, $F(tx + (1-t)y) \subset tF(x) + (1-t)F(y) + C$.

(iii) F is said to be natural quasi-C-concave if for any x, $y \in D$, $t \in [0, 1]$, $z = tx + (1-t)y$, $w \in F(z)$, there exist $\mu \in [0, 1]$ and $w_1 \in F(x)$, $w_2 \in F(y)$ such that $w \in \mu w_1 + (1-\mu)w_2 + C$.

(iv) G is said to be monotone if for any x, $y \in D$, $G(x, y) + G(y, x) \subset -C$.

REMARK 2. Obviously, if F is C-concave, then it is natural quasi-C-concave.

DEFINITION 5([5]). Let D be a nonempty convex subset of a vector space X. A set-valued mapping $F : D \longrightarrow 2^X$ is called KKM-mapping if for each finite subset $\{x_1, \ldots, x_n\} \subset D$, $\mathrm{co}(x_1, \ldots, x_n) \subset \bigcup_{i=1}^n F(x_i)$, where $\mathrm{co}(E)$ is the convex hull of a set E.

FAN LEMMA ([5]). *Let X be a Hausdorff topological vector space, and let D be a nonempty, convex subset of X. Let $F : D \longrightarrow 2^X$ be a KKM-mapping. If all the sets $F(x)$ are closed in X, and if one is compact, then $\bigcap_{x \in D} F(x) \neq \emptyset$.*

3. Existence of solutions of VEP

In this section, using the methods of [4, 11], we prove existence theorems for VEP. Throughout this section, let X and Y be real locally convex Hausdorff spaces, D a nonempty,convex closed subset of X, and $C \subset Y$ a pointed, closed convex cone with apex at the origin and $\mathrm{int}\, C \neq \emptyset$.

LEMMA 4. *Let G, $H : D \times D \longrightarrow 2^Y$ be set-valued mappings satisfing the following conditions:*

(i) $\forall x \in D$, $0 \in G(x, x) \subset C$, $0 \in H(x, x) \subset C$;

(ii) *G is monotone*;
(iii) *for any fixed* x, $y \in D$, *the mapping* $g(t) = G(ty + (1-t)x, y)$, $0 \leqslant t \leqslant 1$, *is* (-$C$)-l.s.c. at $t = 0$;
(iv) *for any fixed* $x \in D$, $G(x, .), H(x, .) : D \longrightarrow 2^Y$ *are C-convex*.

Then the following statements are equivalent:
(I) $\bar{x} \in D$, $G(y, \bar{x}) - H(\bar{x}, y) \subset Y \backslash \mathrm{int}\, C$, $\forall y \in D$;
(II) $\bar{x} \in D$, $G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\mathrm{int}\, C)$, $\forall y \in D$.

Proof. (I) $\Longrightarrow$ (II). For any fixed $y \in D$, $\forall t \in (0, 1]$, $x_t = ty + (1-t)\bar{x} \in D$, by (I), we have

$$G(x_t, \bar{x}) - H(\bar{x}, x_t) \subset Y \backslash \mathrm{int}\, C. \tag{2}$$

By the condition (i) and the C-convexity of $G(x, .)$ and $H(x, .)$, we have

$$tG(x_t, y) + (1-t)G(x_t, \bar{x}) \subset G(x_t, x_t) + C \subset C + C = C, \tag{3}$$

$$tH(\bar{x}, y) \subset tH(\bar{x}, y) + (1-t)H(\bar{x}, \bar{x}) \subset H(\bar{x}, x_t) + C. \tag{4}$$

By (3) and (4), we get

$$tG(x_t, y) + t(1-t)H(\bar{x}, y) \subset -(1-t)G(x_t, \bar{x}) + (1-t)H(\bar{x}, x_t) + C. \tag{5}$$

We claim that

$$G(x_t, y) + (1-t)H(\bar{x}, y) \subset Y \backslash (-\mathrm{int}\, C), \quad \forall t \in (0, 1]. \tag{6}$$

Indeed, if (6) is false, then there exist some $t \in (0, 1]$ and some $a \in G(x_t, y)$, $b \in H(\bar{x}, y)$ such that

$$a + (1-t)b \in -\mathrm{int}\, C. \tag{7}$$

By (5), there exist $z \in G(x_t, \bar{x})$, $w \in H(\bar{x}, x_t)$ and $\bar{c} \in C$ such that $t[a + (1-t)b] = -(1-t)(z-w) + \bar{c}$. By (7), we have

$$(1-t)(z-w) = -t[a + (1-t)b] + \bar{c} \in \mathrm{int}\, C + \bar{c} \subset \mathrm{int}\, C.$$

Hence, $z - w \in \mathit{int}\, C$, which contradicts (2).

Let $g(t) = G(x_t, y) + (1-t)H(\bar{x}, y)$, $t \in [0, 1]$. By the condition (iii), $g(t)$ is (-C)-l.s.c. at $t = 0$. It follows from Lemma 3 that $g(0) \subset Y \backslash (-\mathrm{int}\, C)$, i.e.,

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\mathrm{int}\, C), \quad \forall y \in D.$$

(II) $\Longrightarrow$ (I). In contrary, assume that (I) is false. Then $\exists \bar{y} \in D$ such that

$$G(\bar{y}, \bar{x}) - H(\bar{x}, \bar{y}) \not\subset Y \backslash \mathrm{int}\, C.$$

Therefore, there exist $z \in G(\bar{y}, \bar{x})$ and $w \in H(\bar{x}, \bar{y})$ such that

$$z - w \in \mathrm{int}\, C. \tag{8}$$

On the other hand, since G is monotone, we have

$$G(\bar{y}, \bar{x}) \subset -C - G(\bar{x}, \bar{y}). \tag{9}$$

By (9) and (II), we have

$$z - w \in G(\bar{y}, \bar{x}) - H(\bar{x}, \bar{y}) \subset -C - G(\bar{x}, \bar{y}) - H(\bar{x}, \bar{y}) \subset -C - (Y \backslash (-\mathrm{int}\, C)).$$

Hence, there exist $c \in C$ and

$$y \in Y \backslash (-\mathrm{int}\, C) \tag{10}$$

such that $z - w = -c - y$. By (8), we have

$$-y = c + (z - w) \in c + \text{int}\, C \subset \text{int } C, \ , \text{i.e., } y \in -\text{int } C,$$

which contradicts (10). Thus (I) holds. □

THEOREM 1. *Let G and H be as in Lemma 4, and let all the conditions* (i)–(iv) *of Lemma 4 hold. In addition, G and H satisfy the following conditions*:

(v) *for any fixed* $x \in D$, $G(x, y)$ *is C-l.s.c. in y*; *for any fixed* $y \in D$, $H(x, y)$ *is (-C)-l.s.c. in x*;

(vi) *there is a nonempty, convex compact subset* $K \subset D$ *such that*, $\forall x \in K \backslash core_D K, \exists a \in core_D K,\ G(x, a) + H(x, a) \not\subset Y \backslash (-C)$.

Then $\exists \bar{x} \in K$ *such that*

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\text{int } C), \quad \forall y \in D;$$

In addition, *if C satisfies the condition* (Δ), *then*

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-C \backslash \{0\}), \quad \forall y \in D.$$

First, we prove the following lemmas.

LEMMA 5. *Let all the conditions in Theorem 1 hold. Then there is an* $\bar{x} \in K$ *such that*

$$G(y, \bar{x}) - H(\bar{x}, y) \subset Y \backslash \text{int } C, \quad \forall y \in K.$$

Proof. Define $F{:}K \longrightarrow 2^K$ by

$$F(y) = \{x \in K{:} \quad G(y, x) - H(x, y) \subset Y \backslash \text{int } C\}, \quad \forall y \in K.$$

For any fixed $y \in D$, since $G(y, .) - H(., y)$ is C-l.s.c., by Lemma 2, $F(y)$ is closed in K. We shall show that F is a KKM-mapping.

Suppose it is false. Then there exist $\{y_1, \ldots, y_n\} \subset K$ and $t_1, \ldots, t_n > 0$, $\sum_{i=1}^n t_i = 1$ and $z = \sum_{i=1}^n t_i y_i$ such that $z \notin \bigcup_{i=1}^n F(y_i)$. Then, for each i, $G(y_i, z) - H(z, y_i) \not\subset Y \backslash \text{int } C$, i.e., $\exists a_i \in G(y_i, z),\ b_i \in H(z, y_i)$ such that $a_i - b_i \in \text{int } C$. Thus,

$$\sum_{i=1}^n t_i(a_i - b_i) \in \text{int } C. \tag{11}$$

On the other hand, since G is monotone, we have

$$G(y_i, z) \subset -C - G(z, y_i), \quad \forall i.$$

Since $G(x, y)$ is C-convex in y, it follows from the above that

$$\sum_{i=1}^n t_i G(y_i, z) \subset -C - \sum_{i=1}^n t_i G(z, y_i) \subset -C - G(z, z) - C = -C. \tag{12}$$

Since $H(x, y)$ is C-convex in y, we get

$$\sum_{i=1}^n t_i H(z, y_i) \subset H(z, z) + C \subset C + C = C. \tag{13}$$

By (12) and (13), we have

$$\sum_{i=1}^{n} t_i a_i - \sum_{i=1}^{n} t_i b_i \in \sum_{i=1}^{n} t_i G(y_i, z) - \sum_{i=1}^{n} t_i H(z, y_i) \subset -C. \tag{14}$$

By (11) and (14), we have

$$\sum_{i=1}^{n} t_i a_i - \sum_{i=1}^{n} t_i b_i \in (-C) \cap (\text{int } C) = \emptyset,$$

a contradiction. Hence, F is a KKM-mapping. Since K is compact, by the well-known Fan lemma, there exists $\bar{x} \in \bigcap_{y \in K} F(y)$, i.e.,

$$G(y, \bar{x}) - H(\bar{x}, y) \subset Y \backslash int\ C, \quad \forall y \in K.$$

□

LEMMA 6. *Let D, K and C be as in Theorem 1, and let $\psi : D \longrightarrow 2^Y$ be given. Assume that*

(i) *ψ is C-convex;*

(ii) $x_0 \in \text{core}_D K$, $\psi(x_0) \not\subset Y \backslash (-C)$;

(iii) $\forall y \in K$, $\psi(y) \subset Y \backslash (-\text{int } C)$.

Then $\psi(y) \subset Y \backslash (-\text{int } C)$, $\forall y \in D$.

Proof. Suppose that there is a point $\bar{y} \in D \backslash K$ such that $\psi(\bar{y}) \not\subset Y \backslash (-\text{int } C)$. Then there is $w \in \psi(\bar{y})$ such that $w \in -\text{int } C$. By condition (ii), we have $u \in \psi(x_0)$ such that $u \in -C$. For each $z \in (x_0, \bar{y}]$, $z = tx_0 + (1-t)\bar{y}$, $t \in [0, 1)$, it follows from the C-convexity of ψ that

$$tu + (1-t)w \in t\psi(x_0) + (1-t)\psi(\bar{y}) \subset \psi(z) + C.$$

Therefore, there exist $v \in \psi(z)$ and $c \in C$ such that $tu + (1-t)w = v + c$. Thus, we have

$$v = -c + tu + (1-t)w \in -c - C - \text{int } C \subset -\text{int } C.$$

Hence,

$$\psi(z) \not\subset Y \backslash (-\text{int } C). \tag{15}$$

Since $x_0 \in \text{core}_D K$, we have a point $\bar{z} \in (x_0, \bar{y}] \cap K$. By (15), we get

$$\psi(\bar{z}) \not\subset Y \backslash (-\text{int } C),$$

which contradicts condition (iii). □

Proof of Theorem 1. By Lemma 5, $\exists \bar{x} \in K$ such that

$$G(y, \bar{x}) - H(\bar{x}, y) \subset Y \backslash \text{int } C, \quad \forall y \in K.$$

It follows from Lemma 4 that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\text{int } C), \forall y \in K. \tag{16}$$

Let $\psi(y) = G(\bar{x}, y) + H(\bar{x}, y)$, $\forall y \in D$. Then ψ is C-convex. By (16), we have

$$\psi(y) \subset Y \backslash (-\text{int } C), \quad \forall y \in K.$$

If $\bar{x} \in \text{core}_D K$, then choose $x_0 = \bar{x}$; if $\bar{x} \in K \backslash \text{core}_D K$, by condition (vi) of

Theorem 1, choose $x_0 = a$. We have $x_0 \in \mathrm{core}_D K$ and $\psi(x_0) \not\subset Y\backslash(-C)$. Lemma 6 yields $\psi(y) \subset Y\backslash(-\mathrm{int}\ C), \forall y \in D$, i.e.,

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y\backslash(-\mathrm{int}\ C), \quad \forall y \in D.$$

In addition, if C satisfies the condition (Δ), then there is a pointed, closed convex cone $\widetilde{C}$ of Y such that $C\backslash\{0\} \subset \mathrm{int}\ \widetilde{C}$. Then D,K,$\widetilde{C}$,G and H satisfy all the conditions of this theorem. Applying the proof of the first part for D,K,$\widetilde{C}$,G and H, we conclude that there is $\bar{x} \in K$ such that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y\backslash(-\mathrm{int}\ \widetilde{C}), \quad \forall y \in D.$$

Since $C\backslash\{0\} \subset \mathrm{int}\widetilde{C}$, it follows

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y\backslash(-C\backslash\{0\}), \quad \forall y \in D.$$ □

REMARK 3. (i) If D is a compact convex subset, then the condition (vi) is satisfied vacuously with $K = D$, since then $K\backslash core_D K = \emptyset$. (ii) If G, H are vector single-valued mappings, then we get Theorem 3.1 of [11]. If G, H are scalar single-valued functions, then we get Theorem 1 of [4].

Let $L(X, Y)$ be the space of all continuous linear operators from X into Y. For any $l \in L(X, Y)$, $\langle l, x\rangle$ denotes the evaluation of l at $x \in X$. Let $T : D \longrightarrow 2^{L(X,Y)}$ be given, and $\langle T(x), y\rangle = \bigcup_{l\in T(x)}\langle l, y\rangle$.

DEFINITION 6. Let T be as above. T is said to be monotone if, $\forall x, y \in D$, $\langle T(x), x - y\rangle + \langle T(y),\ y - x\rangle \subset -C$.

COROLLARY. *Let $T : D \to 2^{L(X,Y)}$ be a set-valued mapping and $h : D \to Y$ be a single-valued mapping satisfying the following conditions:*

(i) *T is monotone*;
(ii) *for any fixed $x, y \in D$, $g(t) = \langle T(ty + (1 - t)x),\ x - y\rangle$, $0 \leqslant t \leqslant 1$, is* (-*C*)-*l.s.c. at $t = 0$*;
(iii) *h is C-convex and continuous;*
(iv) *there is a nonempty*, *compact convex subset K of D such that*, $\forall x \in K\backslash\mathrm{core}_D K$, $\exists a \in \mathrm{core}_D K$,
$\langle T(x),\ x - a\rangle + h(a) - h(x) \not\subset Y\backslash(-C)$.

Then there is $\bar{x} \in K$ such that

$$\langle T(\bar{x}), \quad \bar{x} - y\rangle + h(y) - h(\bar{x}) \subset Y\backslash(-\mathrm{int}\ C), \quad \forall y \in D.$$

Proof. In Theorem 1, choose $G(x, y) = \langle T(x),\ x - y\rangle$ and $H(x, y) = h(y) - h(x)$. Then Theorem 1 yields the conclusion. □

THEOREM 2. *Let D, C, G and H be as in Theorem 1*, *and let the conditions* (i)–(v) *of Theorem 1 hold. In addition*, *assume that the following condition holds:*

(vi)' *there exists a nonempty, compact convex subset* $B \subset D$ *such that,* $\forall x \in D \backslash B$, $\exists a \in B$,

$$G(a, x) - H(x, a) \not\subset Y \backslash \text{int } C. \tag{17}$$

Then there exists $\bar{x} \in B$ *such that*

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\text{int } C), \quad \forall y \in D.$$

If C satisfies the condition (Δ), *then*

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-C \backslash \{0\}), \quad \forall y \in D.$$

Proof. For any $y \in D$, put

$$F(y) = \{x \in B : G(y, x) - H(x, y) \subset Y \backslash \text{int } C\}.$$

Then F has the finite intersection property. In fact, let $\{y_1, \ldots, y_n\} \subset D$, and let $K = \text{co}(B \cup \{y_1, \ldots, y_n\})$. Then K is a compact convex subset of D. Applying the argument of Lemma 5, there is $\tilde{x} \in K$ such that

$$G(y, \tilde{x}) - H(\tilde{x}, y) \subset Y \backslash \text{int } C, \quad \forall y \in K. \tag{18}$$

If $\tilde{x} \in K \backslash B$, then by the condition (vi)', $\exists a \in B$ such that

$$G(a, \tilde{x}) - H(\tilde{x}, a) \not\subset Y \backslash \text{int } C,$$

a contradiction to (18). We conclude $\tilde{x} \in B$. Hence, $\tilde{x} \in \bigcap_{i=1}^{n} F(y_i)$, i.e., $F(y)$ has the finite intersection property. Since B is compact, we have $\bar{x} \in \bigcap_{y \in D} F(y)$. Thus, $\bar{x} \in B$ and

$$G(y, \bar{x}) - H(\bar{x}, y) \subset Y \backslash \text{int } C, \quad \forall y \in D.$$

By Lemma 4, it follows that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\text{int } C), \quad \forall y \in D.$$

The second part can be proven similarly to the second part of Theorem 1.

□

REMARK 4. (i) From the proof of Theorem 2, we have the following fact: if $\bar{x}$ is a solution of (VEP 1) or (VEP 3) in Theorem 2, then $\bar{x} \in B$. Indeed, let $\bar{x}$ is a solution of (VEP 1). By Lemma 4, $\bar{x}$ also satisfies

$$(**) \qquad G(y, \bar{x}) - H(\bar{x}, y) \subset Y \backslash \text{int } C, \quad \forall y \in D.$$

If $\bar{x} \in D \backslash B$, then by condition (vi)' of Theorem 2, $\exists a \in B$ such that

$$G(a, \bar{x}) - H(\bar{x}, a) \not\subset Y \backslash \text{int } C,$$

which contradicts (**). Hence, $\bar{x} \in B$. For (VEP 3), the argument is similar.

(ii) If G and H are vector single-valued mappings, from the above Theorem 2, we obtain Theorem 3.7 of [11].If G, H are scalar single-valued functions, we get Theorem 1A of [4].We drop the maximal monotonicity of G.

4. Convexity of solution sets of VEP

THEOREM 3. *Let* X, Y, D *and* C *be as in Theorem 1, and let* $C^\sharp \neq \emptyset$. *Let* $G, H : D \times D \longrightarrow 2^Y$ *be set-valued mappings satisfying the conditions* (i)–(vi) *in Theorem 1. In addition, assume that* G, H *satisfy the following condition:*

(vii) *for any fixed* $y \in D$, $G(.,y) + H(.,y) : D \longrightarrow 2^Y$ *are natural quasi-C-concave*;

Then the solution set of (*VEP 1*)

$$E = \{\bar{x} \in K : G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-int\ C), \forall y \in D\}$$

is convex. If C satisfies the condition (Δ), *then the solution set of* (*VEP 3*)

$$E_* = \{\bar{x} \in K : G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-C \backslash \{0\}), \quad \forall y \in D\}$$

is convex.

Proof. We only show the case of (VEP 1). The other one can be proven similarly. By Theorem 1, $E \neq \emptyset$. $\forall f \in C^\sharp$, let $T(f) = \{\bar{x} \in K : f[G(\bar{x}, y) + H(\bar{x}, y)] \geq 0, \ \forall y \in D\}$, then $E = \bigcap_{f \in C^\sharp} T(f)$, where $f[G(\bar{x}, y) + H(\bar{x}, y)] \geqslant 0$ means that $\forall z \in G(\bar{x}, y)$, $w \in H(\bar{x}, y)$, $f(z + w) \geqslant 0$.

Indeed, let $\bar{x} \in E$. Then $G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\text{int } C)$, $\forall y \in D$. If $f \in C^\sharp$, for any $z \in G(\bar{x}, y)$, $w \in H(\bar{x}, y)$, $z + w \notin -\text{int } C$ and $f(z + w) \geqslant 0$. Hence, $\forall f \notin C^\sharp$ $\bar{x} \notin T(f)$. Then, $E \subset \bigcap_{f \in C^\sharp} T(f)$.

On the other hand, if $\bar{x} \in \bigcap_{f \in C^\sharp} T(f)$, then $\forall f \in C^\sharp$ such that $\bar{x} \in T(f)$, that is,

$$f(G(\bar{x}, y) + H(\bar{x}, y)) \geqslant 0, \quad \forall y \in D. \tag{19}$$

We claim that $\bar{x} \in E$, that is,

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\text{int } C), \quad \forall y \in D.$$

Suppose to the contrary that there is a $\bar{y} \in D$ such that

$$G(\bar{x}, \bar{y}) + H(\bar{x}, \bar{y}) \not\subset Y \backslash (-\text{int } C).$$

Then there are $z \in G(\bar{x}, \bar{y})$, $w \in H(\bar{x}, \bar{y})$ such that $z + w \in -\text{int } C$. Since $f \in C^\sharp$, it follows $f(z + w) < 0$, which contradicts (19). Hence, $\bar{x} \in E$.

Next, we show that $\forall f \in C^\sharp$, $T(f)$ is a nonempty,convex subset of K.

Let $f \in C^\sharp$ be given. Let x_1, $x_2 \in T(f)$, and $\bar{x} = tx_1 + (1 - t)x_2$, $0 \leqslant t \leqslant 1$. Then we have

$$f(G(x_i, y) + H(x_i, y)) \geqslant 0, \quad \forall y \in D, \quad i = 1, 2. \tag{20}$$

For any fixed $y \in D$, by condition (vii), $\forall z \in G(\bar{x}, y) + H(\bar{x}, y)$, there are $z_i \in G(x_i, y) + H(x_i, y)$, $i = 1, 2$, $\mu \in [0, 1]$ such that

$$z \in \mu z_1 + (1 - \mu) z_2 + C.$$

Then, there is some $\bar{c} \in C$ such that

$$z = \mu z_1 + (1 - \mu) z_2 + \bar{c}. \tag{21}$$

By (20) and (21), we have $f(z) \geqslant 0$. Since z is arbitrary, we get $\bar{x} \in T(f)$, i.e., $T(f)$ is convex. Since $E = \bigcap_{f \in \sharp} T(f)$, E is convex. □

Using a similar proof, we have the following result.

THEOREM 4. *Let X, Y, D and C be the same as in Theorem 3. Let G, $H : D \times D \longrightarrow 2^Y$ satisfy all the conditions in Theorem 2. In addition, assume that the following condition holds:*

(vi) *for any fixed* $y \in D$, $G(.,y) + H(.,y) : D \longrightarrow 2^Y$ *is natural quasi-C-concave.*

Then the solution set of (VEP 1)

$$E = \{\bar{x} \in B : G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-\text{int } C), \quad \forall y \in D\}$$

is convex. If C satisfies condition (Δ), *then the solution set of (VEP 3)*

$$E_* = \{\bar{x} \in B : G(\bar{x}, y) + H(\bar{x}, y) \subset Y \backslash (-C \backslash \{0\}), \quad \forall y \in D\}$$

is convex.

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References

1. Ansari, Q.H., Oettle, W. and Schläger, D.(1997), A generalization of vector equilibria. *Mathematical Methods of Operations Research* 46, 147–152.
2. Berge, C. (1963), *Topological Spaces*, Oliver and Boyd Ltd., Edinburgh and London.
3. Bianchi, M., Hadjisavvas, N. and Schaible, S. (1997), Vector equilibrium problems with generalized monotone bifunctions. *Journal of Optimization Theory and Applications* 92, 527–542.
4. Blum, E. and Oettle, W. (1994), From optimization and variational inequalities to equilibrium problems. *The Mathematics Student* 63, 123–145.
5. Dugundji, J. and Granas, A. (1982), *Fixed Point Theory* (*Volume 1*), PWN-Polish Scientific Publishers Warszawa.
6. Fu, J.Y. (2000), Generalized vector quasi-equilibrium problems. *Mathematical Methods of Operations Research* 52, 57–64.
7. Giannessi, F. (ed.) (2000), *Vector Variational Inequalities and Vector Equilibria*, Kluwer Academic Publishers, Dordrecht, Boston, London.
8. Jahn, J. (1986), *Mathematical Vector Opimization in Partially Ordered Linear Spaces*, Verlag, Peter Lang Frankfurt am Main, Bern, New York.
9. Luc, D.T. (1989), Theory of Vector Optimization. In: Lecture Notes in Economics and Mathematical Systems, Vol. 319, Springer-Verlag Berlin.
10. Song, W. Vector Equilibrium Problems with Set-valued Mappings (in the above reference [7], pp. 403–422).
11. Tan, N.X. and Tinh, P.N. (1998), On the existence of equilibrium points of vector functions. *Numerical Functional Analysis and Optimization* 19, 141–156.
12. Tanaka, T. (1994), Generalized quasiconvexities, cone saddle points, and minimax theorem for vector valued functions. *Journal of Optimization Theory and Applications* 81, 355–377.